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Radial Velocity Spectrometers: What They Measure

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Lincoln Laboratory

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

LEXINGTON, MASSACHUSETTS

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FOR THE COMMANDER

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ABSTRACT

This report provides the theoretical analysis, within the framework of special relativity, for the information content of a radial velocity spectrometer. This device is an automated telescope-photometer-spectrometer combination. It measures the displacement between a Doppler-shifted set of spectral lines and its at-rest version. Thus, it provides a measurement of "radial velocity." Exactly what kinematical content is revealed by such a measurement within the solar system is brought out herein. The discussion is couched in terms of a satellite-based sensor observing another orbiting body. Both objects are presumed to orbit about the Earth. Sufficient generality is presented to treat any other case also. In the specific case of interest, both the topocentric distance and the topocentric radial velocity are present in the measured quantity.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

RADIAL VELOCITY SPECTROMETERS: WHAT THEY MEASURE

I. INTRODUCTION

Passive optical measurements of artificial satellites and ballistic missiles are usually thought of as providing angles-only information; that is, such observations can only provide the direction to (or position of) a source of light. As should be well known from the astronomical literature, much more can be learned about the light source's nature, center of mass motion(s), and dynamical state. In particular, by observing the shift of various spectral lines from their at-rest locations one can determine the radial velocity of the source. For stars and galaxies the nonrelativistic approximation may be written as

$$\Delta\lambda / \lambda = v_r / c$$

wherein $\Delta\lambda$ is the signed wavelength shift at wavelength λ (i.e., the Doppler shift), c is the speed of light in vacuo, and v_r is the signed component of the projection of the source's velocity vector along the line of sight to the source (i.e., the radial velocity). The physics of the Doppler shift applies equally well to luminous bodies and to those visible owing to reflected light. Thus, in principle, artificial satellites too may have their Doppler shifts measured by passive, optical means.

However, in the case of artificial satellites, the dynamical situation is a little more complex than it is for stars or galaxies. Indeed, the above formula is an oversimplification for these celestial objects too as the telescope is (usually) on the surface of a rotating Earth, the Earth's center of mass is in motion about the Earth-Moon barycenter, the Earth-Moon barycenter is in motion about the Sun, and the Sun is in motion about the center of mass of the solar system. The solar system barycentric reference frame is the best approximation that we have to a realizable inertial frame. (For galaxies one should add yet other components, as the solar system barycenter is in motion relative to the nearby stars and the center of the Milky Way.)

All the corrections necessary owing to the conflict between the natural reference frame of the observations and the desire to be in an inertial reference frame are very small and simple to perform nonrelativistically. The typical precision in galaxy work is $\simeq 50$ km/s, while for stellar work 1 km/s is considered good. Since the entire range of satellite speeds is only $\simeq 20$ km/s, a much higher precision is required in order to be a useful addition to the endeavors of initial orbit determination, rapid maneuver detection, or to the highly refined differential correction of orbits. This not only puts severe demands on the instrumentation, but requires a more sophisticated analysis of exactly what is measured. That analysis is contained in this Report.

Before getting to the analysis, a few words about the potential instrument are in order. The classical astronomical technique of radial velocity determination utilizes a spectrograph to photograph the dispersed light from the star and then to measure the line shifts after developing the

photographic plate. This is a slow and very time-consuming process, especially as all the light from the source is being used to ascertain only one quantity. Moreover, this single number represents a global aspect of the entire spectrum. Thus, first Fellgett* and then, especially, Griffin† proposed utilizing a cross-correlation technique at the telescope to increase the efficiency of the measurement. Such automated devices are now known as *radial velocity spectrometers*. Their speed of operation is such that a precision of $\simeq 10$ m/s on a 10^{m} object during an integration time of $\simeq 10$ s seems feasible. Future reports shall dwell on the design of such a device keyed to the solar spectrum.

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* P.B. Fellgett, Optica Acta **2**, 9 (1953).

† R.F. Griffin, Astrophys. J. **148**, 465 (1967).

II. COMPLICATIONS

The artificial satellites or ballistic missiles that one might wish to observe with a radial velocity spectrometer shine by reflected sunlight. Therefore, the shape of the line spectrum of the Sun's light, the temporal variation of these features, and the nature of the continuum's envelope are of great importance. The Sun can be regarded as a point source; thus, such problems are greatly alleviated because one is effectively integrating over the whole $\simeq 32'$ solar disk. This washes out much of the rapid temporal and small-scale spatial variability. Also, the Sun is a very well studied object. This fund of knowledge can be utilized to minimize data reduction problems. In particular, one can concentrate on that portion of the solar spectrum arising from the more stable photosphere rather than from the less stable chromosphere or corona. A more detailed discussion of these points will appear in a future Report, and herein I shall merely quote the essential point; microturbulence in the solar atmosphere occurs on the $\simeq 3$ m/s level on time scales short enough to limit one's ultimate precision to this. Note that this is about a part in ten thousand of the quantity we wish to measure. It will turn out that many different aspects of this measurement converge at a level of 1 in 10^4 and, therefore, I shall not aim at any higher precision in the final formulas.

A second source of complications is the rotational motions of the reflector. Any such rotation will have the effect of broadening the Sun's spectral features owing to the opposite Doppler shifts of the leading and trailing edges of the satellite. Of course, the line spectrum comes to us already broadened owing to the Sun's 25-day rotation, atmospheric micro- and macro-turbulence, other large scale convective motions, and so on. Theoretically, the additional broadening owing to a satellite's rotation could be used to compute the responsible angular velocity. This topic will not be discussed herein except to note that all of the solar lines will be equally broadened by such motions. Their relative locations will not shift nor will their absolute locations. Hence, the measurement of the satellite's center of mass Doppler shift is not directly affected by a rotation (or rotations) about the center of mass.

There are two other reflector-induced complications. The first has to do with the nonuniform spectral reflectivity of the satellite's surface and the second has to do with its nonuniform topography. Because the satellite will not be a diffuse, white, reflecting sphere, or even a diffuse, grey, reflecting solid, different portions of the solar spectrum will be emphasized after reflection. That is, the gradient of the reflected continuum will not match that of the Sun itself. Were we attempting to utilize intensity information, then this would be a severe problem. We are not, however, and such varying spectral response in the reflection coefficient cannot alter the detected Doppler shift. Furthermore, for practical reasons of instrumental design, the whole visible wavelength band from 4000-7000 Å will not be utilized. Instead, we shall probably choose a several-hundred Ångstrom stretch (or several such disjoint stretches). This will also serve to minimize the effects of a slowly varying background level or a change in the gradient of the background.

The final reflector-related complication comes about because of topographical irregularities on the satellite's surface. These, coupled with the small apparent angular size of the solar disc,

can lead to specular reflections. Such reflections are very bright and of a very short temporal duration. If the radial velocity spectrometer is using a time averaged measurement technique (e.g., scanning the spectrum), then the presence of a very short burst of very intense signal will cause the implicit weighting of the exposure's duration to be highly nonuniform. This will result in a systematic error as the epoch of the Doppler shift will not be appropriate owing to the intensity averaging. Proper instrumental design and data reduction will minimize this effect.

Finally, it is necessary to discuss sensor-related complications. If the spectrometer is onboard a spacecraft, then the perceived Doppler shift obviously includes a component owing to the sensor's motion. As discussed above, this type of complication is routinely handled in the astronomical case. What makes the problem more complicated herein is the use to which the data will be put and one's desire to have the information content simply relate to some aspect of the dynamical state of the reflecting satellite. If the instrument is ground-based, then this complication is present too. Of more importance may be the additional corruption of the solar spectrum owing to winds in the atmosphere and small, localized refractive elements of air that eventually build up the seeing disc.

Complications owing to the Earth's atmosphere are considerable but do not affect, in principle, the measurement itself. Similarly, a time dependent interplanetary plasma will affect the propagation of the light ray from the Sun to the satellite and thence to the sensor. The incorporation of these departures from empty space and geodesic paths represent minor data reduction problems that I acknowledge but will treat in depth after the instrument is designed.

III. THE STRUCTURE OF THE ANALYSIS

It is much more reliable, and esthetically pleasing, to conduct an analysis such as this with the most complete physics *ab initio* and at the most rigorous mathematical level possible. Because photons are inherently relativistic objects, and gravitational fields are involved, I might declare general relativity to be the appropriate venue. A little thought shows that this represents considerable overkill, as the largest general relativistic effect is to cause an additional redshift owing to the gravitational potential difference between the target satellite or missile and the sensor. If ΔU is the amplitude of the potential difference, then the additional redshift is approximately equal to $\Delta U/c$. This is $\simeq GM_E/cR_E$ or of order 10^{-4} km/s. Hence, it can be safely neglected compared to the desired precision.

As special relativity is the next step down from general relativity, and also represents the natural covariance group of Maxwell's equations, this is the physical framework in which the analysis shall be carried out. The Earth's orbital speed is $\simeq$ 30 km/s or a part in 10^4 ($c = 3 \times 10^5$ km/s). Similarly, satellite or sensor speeds can not be any higher (or much lower) in order of magnitude. Therefore, an eventual simplification of the fully covariant results to first order will be permissible and confluent with the best attainable precision (owing to imperfect knowledge of the fine scale temporal structure of the Sun, not to the technique; it is also true that if we tried to achieve centimeters/second or better precision that instrumental problems would rapidly dominate the practical development). Finally, the last small quantity of the problem is the solar parallax (i.e., the sensor to satellite vs sensor or satellite to solar distance ratio). This is $\simeq 8''.8$, or again a part in ten thousand.

The nature of the problem is almost clear then. We commence with a photon of wavelength λ, frequency ν emitted from the solar photosphere. This photon travels on a straight line at constant speed c until it reaches the surface of some artificial satellite or ballistic missile of interest (the "target"). The photon is reflected from the satellite's surface and travels toward the sensor. As perceived by the sensor, it has frequency ν' and wavelength λ'. By comparing λ' to the wavelength of a similar photon which traveled directly to the sensor, we obtain the Doppler shift. I shall call the wavelength of this second photon λ'' (frequency ν''). There are at least eight (8!) different coordinate systems hidden in the above description. Moreover, it is not apparent that λ' should be compared with λ'' as opposed to, for instance, the value of λ a sensor at the Earth-Moon barycenter would detect, averaged over the course of a tropical year. Once the proposed analysis is fully worked out, modifications of this nature will be relatively simple to tack on.

IV. LORENTZ TRANSFORMATIONS

Let π be any contravariant four-vector. Let p_0 be its time-like part and $\underline{p}$ be its space-like part, $\pi = (p_0, \underline{p})$. An observer in the inertial reference frame I measures π. Consider a different observer in the inertial reference frame I′. The observer in I describes the motion of the primed reference frame as being on a straight line with constant velocity $\underline{u}$. What will the observer in I′ detect for π? The answer is given by a Lorentz transformation with parameter $\underline{\beta}$, $\underline{\beta} = \underline{u}/c$, $(\beta = |\underline{\beta}|)$, namely the observer at rest in I′ would measure $\pi' = (p'_0, \underline{p}')$ where

$$
\begin{aligned}
p'_0 &= \gamma(p_0 - \underline{p} \cdot \underline{\beta}) \\
p'_{||} &= \gamma(p_{||} - \beta p_0) \\
\underline{p}'_{\perp} &= \underline{p}_{\perp} \quad .
\end{aligned}
\tag{1a}
$$

Herein, $\gamma = (1 - \beta^2)^{-1/2}$ and the parallel and perpendicular subscripts refer to components along and orthogonal to $\underline{u}$. As $\underline{p} \cdot \underline{u}/u$ is the component of $\underline{p}$ along $\underline{u}$, it follows that $p_{||} = \underline{p} \cdot \underline{u}/u$ or $\underline{p}_{||} = (\underline{p} \cdot \underline{u})\underline{u}/u^2$. Consequently, one may write $\underline{p}_{\perp}$ as $\underline{p} - \underline{p}_{||}$ and a vector form of the Lorentz transformation as

$$\pi' = \Big(\gamma(p_0 - \underline{p} \cdot \underline{\beta}),\ \underline{p} + [(\gamma - 1)\, \underline{p} \cdot \underline{u}/u^2]\, \underline{u} - \gamma p_0 \underline{\beta}\Big) \quad . \tag{1b}$$

Because $(\underline{p} \cdot \underline{u}/u^2)\, \underline{u} = (\underline{p} \cdot \underline{\beta}/\beta^2)\, \underline{\beta}$, a third form is

$$\pi' = \Big(\gamma(p_0 - \underline{p} \cdot \underline{\beta}),\ \underline{p} + [(\gamma - 1)\, \underline{p} \cdot \underline{\beta}/\beta^2]\, \underline{\beta} - \gamma p_0 \underline{\beta}\Big) \quad . \tag{1c}$$

Equations (1) and the definition of an inertial frame are all the special relativity that I require.

To illustrate Equations (1), I will consider the effects of a Lorentz transformation on two different four-vectors. One is the space-time location vector $\rho = (r_0, \underline{r}) = (ct, \underline{r})$, where t is the proper time and $\underline{r}$ is the usual three-dimensional location, $\underline{r} = (x, y, z)$. The other is the four-dimensional wave vector $\kappa = (k_0, \underline{k})$, where $k_0 = |\underline{k}| = \omega/c$, ω is the angular frequency of a photon of wavenumber k, wavelength $\lambda = 2\pi/k$, propagating in the direction of $\underline{k}$ (frequency $\nu = \omega/2\pi$). Note that the four-dimensional scalar product of ρ and κ, which is equal to $r_0 k_0 - \underline{r} \cdot \underline{k} = \omega t - \underline{r} \cdot \underline{k}$, represents an invariant, namely the phase of the electromagnetic disturbance associated with the photon.

If an observer at rest in I measures ρ and one at rest relative to I′ measures ρ', then

$$
\begin{aligned}
t' &= \gamma(t - \underline{r} \cdot \underline{\beta}/c) \\
\underline{r}' &= \underline{r} + [(\gamma - 1)\, \underline{r} \cdot \underline{\beta}/\beta^2]\, \underline{\beta} - \gamma ct \underline{\beta} \quad .
\end{aligned}
\tag{2}
$$

Subsumed in Equations (2) are the two additional assumptions that at $t = t' = 0$ the origins of I and I′ coincided and that no rotations are necessary to connect I to I′. The arbitrary translations or rotations that might be necessary present only mathematical complexities, not any new physics. From Equations (2) one can derive the special relativistic time-dilation, the Fitzgerald-Lorentz contraction, and so on.

The derivation of the velocity transformation requires a bit more care as $\underline{v} = d\underline{r}/dt$ is not a four-vector. One can add a time-like component to $\underline{v}$ and form a proper four-vector (this new vector is frequently termed the "world velocity"), but I can deduce $\underline{v}' = d\underline{r}'/dt'$ by a straightforward differentiation of Equations (2). The result is

$$\underline{v}' = \frac{\underline{v} + [(\gamma - 1)\,\underline{v}\cdot\underline{u}/u^2]\,\underline{u} - \gamma\underline{u}}{\gamma(1 - \underline{v}\cdot\underline{u}/c^2)} \quad . \tag{3}$$

This formula can be used to compute relative velocities. Consider two particles, labeled 1 and 2, with velocities $\underline{v}_1$ and $\underline{v}_2$ as measured by an observer at rest in some inertial frame. What is the relative velocity of particle 1 with respect to particle 2? It is the result of applying Equation (3) to $\underline{v}_1$ with $\underline{u}$ equal to $\underline{v}_2$. Namely,

$$\underline{v}_1' = \frac{\underline{v}_1 + [(\gamma_2 - 1)\,\underline{v}_1\cdot\underline{v}_2/v_2^2]\,\underline{v}_2 - \gamma_2\underline{v}_2}{\gamma_2(1 - \underline{v}_1\cdot\underline{v}_2/c^2)} \tag{4a}$$

where $\underline{v}_1'$ is the sought-for relative velocity, $\gamma_2 = (1 - \beta_2^2)^{-1/2}$, and $\underline{\beta}_2 = \underline{v}_2/c$. As a final point, a nice form for $|\underline{v}_1'|$ is

$$|\underline{v}_1'| = \frac{[|\underline{v}_1 - \underline{v}_2|^2 - |\underline{v}_1 \times \underline{v}_2|^2/c^2]^{1/2}}{(1 - \underline{v}_1\cdot\underline{v}_2/c^2)} \quad . \tag{4b}$$

Now let us consider the application of a Lorentz transformation to the four-dimensional wave vector $\kappa = (k_0, \underline{k})$. The result is given in Equation (1a),

$$k_0' = \gamma(k_0 - \underline{k}\cdot\underline{\beta}),\ k_{||}' = \gamma(k_{||} - \beta k_0),\ \underline{k}_{\perp}' = \underline{k}_{\perp} \quad .$$

But $k_0' = \omega'/c$ and if I write $\underline{k}$ as $k\underline{n}$, where $\underline{n}$ is a unit vector in the direction of propagation, then the time-like part of $\kappa' = (k_0', \underline{k}')$ implies that

$$\omega' = \gamma\omega(1 - \underline{n}\cdot\underline{\beta}) \quad .$$

This embodies the Doppler shift in that $\omega' \neq \omega$ (or $\lambda' \neq \lambda$).

In order to see the consequences of the space-like part of the transformation, let θ be the angle between $\underline{u}$ and $\underline{n}$. Then, $k_{||} = k\cos\theta$ and $k_{\perp} = k\sin\theta$. Writing $k_{||}' = k'\cos\theta'$ and $k_{\perp}' = k'\sin\theta'$, it follows that

$$k_{||}' = k\gamma(\cos\theta - \beta) \qquad \text{and} \qquad k_{\perp}' = k\sin\theta$$

or, after division,

$$\tan\theta' = \frac{\sin\theta}{\gamma(\cos\theta - \beta)} \quad , \qquad \text{i.e., } \theta \neq \theta' \quad .$$

This shows that the two observers perceive the light ray to be propagated in different directions; i.e., this formula embodies the aberration of light. Note that there is aberration even if the direction of propagation is transverse ($\theta = \pi/2$) to the relative velocity vector. There is a Doppler shift in this circumstance too.

V. FROM THE SUN TO THE SENSOR

Start a heliocentric clock at time t = 0. At this instant, the Sun has some velocity vector relative to the solar system barycentric inertial reference frame. At t = 0 construct an instantaneously inertial reference frame at the Sun's center by a translation of the appropriate amount and via the appropriate Lorentz transformation. Also, at t = 0 the Sun emits photons of wavenumber k, angular frequency ω, isotropically into space. I now focus on the photon which will be reflected by the target.

This reflection occurs at some time t_T (> 0) later, as measured in the original instantaneously inertial, heliocentric reference frame, when the target has a velocity $\underline{v}_T$ as measured in this same reference frame. What wave vector does the target perceive? It sees an incident four-vector κ_{inc}, where κ_{inc} is related to the emitted κ by a Lorentz transformation with velocity $\underline{v}_T$. Thus, with $\gamma_T = (1 - \beta_T^2)^{-1/2}$, $\underline{\beta}_T = \underline{v}_T/c$,

$$\kappa_{inc} = \left(\gamma_T(k_0 - \underline{k} \cdot \underline{\beta}_T),\ \underline{k} + [(\gamma_T - 1)\,\underline{k} \cdot \underline{\beta}_T/\beta_T^2]\,\underline{\beta}_T - \gamma_T k_0 \underline{\beta}_T\right) \quad .$$

It is understood that $\underline{v}_T$ refers only to the target's velocity at time $t = t_T$ as measured in the instantaneously inertial, heliocentric reference frame of time t = 0.

Now consider, in the instantaneously inertial reference frame of the target, what the reflection process looks like. If κ_{inc} has spatial part $\underline{k}_{inc}$ and the reflected wave four-vector κ_{refl} has spatial part $\underline{k}_{refl}$, then these two three-vectors lie in a plane containing the outward surface normal at the point of reflection; see Figure 1. Also, the angle of incidence equals the angle of reflection *in this reference frame.* Thus, I shall be able to compute κ_{refl} as follows: I write $\underline{k}_{inc}$ as

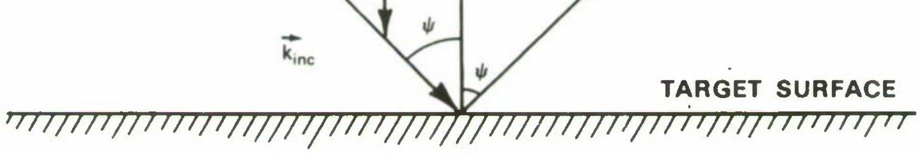

Figure 1. Reflection process in the co-moving reference frame of the reflector. The angle of incidence is denoted by ψ and is also the angle of reflection. The outward surface normal is denoted by $\underline{n}$ and the incoming spatial part of the wave vector is denoted by $\underline{k}_{inc}$. The wave vector is decomposed into parts perpendicular and parallel to $\underline{n}$. $\underline{k}_{\parallel}$ is reversed upon reflection so $\underline{k}_{refl} = -\underline{k}_{\parallel} + \underline{k}_{\perp}$.

$$\underline{k}_{inc} = \underline{k}_{\perp} + \underline{k}_{\parallel}$$

where parallel and perpendicular now refer to the outward unit normal $\underline{n}$. Since

$$\underline{k}_{\parallel} = (\underline{k}_{inc} \cdot \underline{n})\, \underline{n} \qquad \text{and} \qquad \underline{k}_{\perp} = \underline{k}_{inc} - \underline{k}_{\parallel}$$

and what happens upon reflection is that the component orthogonal to the surface is reversed while the component parallel to the surface (i.e., perpendicular to $\underline{n}$) is unchanged, all with no change of frequency, it follows that

$$\underline{k}_{refl} = \underline{k}_{\perp} - \underline{k}_{\parallel} = \underline{k}_{inc} - 2(\underline{k}_{inc} \cdot \underline{n})\, \underline{n} \quad .$$

Whence,

$$\kappa_{refl} = \Big(\gamma_T(k_0 - \underline{k} \cdot \underline{\beta}_T),\ \underline{k}_{inc} - 2(\underline{k}_{inc} \cdot \underline{n})\, \underline{n}\Big) \tag{5a}$$

with

$$\underline{k}_{inc} = \underline{k} + [(\gamma_T - 1)\, \underline{k} \cdot \underline{\beta}_T/\beta_T]\, \underline{\beta}_T - \gamma_T k_0 \underline{\beta}_T \quad . \tag{5b}$$

The reflected photon travels away from the target. It impinges upon the sensor at some time t_S ($>t_T$) as measured in the original, instantaneously inertial, heliocentric reference frame. At that instant, in this same reference frame, the sensor has velocity $\underline{v}_S$. To see what the sensor sees, I must transform κ_{refl}, which is expressed relative to the instantaneously inertial reference frame of the target in Equations (5), into the sensor's reference frame. This implies performing a Lorentz transformation for velocity $\underline{v}'_S$, where $\underline{v}'_S$ is the velocity of the sensor at $t = t_S$ relative to what the target's velocity was at time t_T. Equation (4a) can be used with $\underline{v}_1 = \underline{v}_S$, $\underline{v}_2 = \underline{v}_T$;

$$\underline{v}'_S = \frac{\underline{v}_S + [(\gamma_T - 1)\, \underline{v}_S \cdot \underline{v}_T/v_T^2]\, \underline{v}_T - \gamma_T \underline{v}_T}{\gamma_T(1 - \underline{v}_S \cdot \underline{v}_T/c^2)} \quad , \qquad \underline{\beta}'_S = \underline{v}'_S/c \quad . \tag{6}$$

Applying Equation (1c) to κ_{refl} I obtain the wave vector measured by the sensor,

$$\begin{aligned} \kappa' = \Big(&\gamma'_S \{\gamma_T(k_0 - \underline{k} \cdot \underline{\beta}_T) - \underline{\beta}'_S \cdot [\underline{k}_{inc} - 2(\underline{k}_{inc} \cdot \underline{n})\, \underline{n}]\} \ , \\ &\underline{k}_{inc} - 2(\underline{k}_{inc} \cdot \underline{n})\, \underline{n} + (\gamma'_S - 1)\, \underline{\beta}'_S \cdot [\underline{k}_{inc} - 2(\underline{k}_{inc} \cdot \underline{n})\, \underline{n}]\, \underline{\beta}'_S/(\beta'_S)^2 \\ &- \gamma'_S \gamma_T (k_0 - \underline{k} \cdot \underline{\beta}_T) \underline{\beta}'_S\Big) \end{aligned} \tag{7}$$

where $\gamma'_S = [1 - (\beta'_S)^2]^{-1/2}$. The temporal part of κ' is ω'/c, where ω' is the angular frequency at which the sensor detects the photon originally emitted from the Sun with angular frequency ω. The Doppler-shift analysis involves examining ω'/ω or some similar quantity. I shall make the reason for my deliberate fuzziness clear below. Equation (7) is the penultimate result.

VI. INTERPRETATION

Equation (7) looks sufficiently formidable that it might be wise to try and see what it implies in a simple case. A two-dimensional illustration should suffice and I take $\underline{k} = k(\cos\theta, \sin\theta, 0)$. Suppose that the reflecting satellite is moving along the x axis, $\underline{\beta}_T = (\beta_T, 0, 0)$, and that the sensor is at rest, $\underline{\beta}_S = \underline{0}$. Further suppose that the outwardly directed normal is in the direction of the negative x axis, $\underline{n} = (-1, 0, 0)$. The other quantities that I need are $\underline{\beta}'_S = -\underline{\beta}_T$, $\gamma'_S = \gamma_T$, and $\gamma_S = 1$.

Note that $\underline{\beta}_T \cdot \underline{k} = k\beta_T \cos\theta$, so κ_{inc} has the form

$$\kappa_{inc} = \left(\gamma_T k(1 - \beta_T \cos\theta),\ \underline{k} + k(\gamma_T - 1)\underline{\beta}_T \cos\theta/\beta_T - k\gamma_T\underline{\beta}_T\right) \quad .$$

The spatial part of this is $\underline{k}_{inc}$,

$$\underline{k}_{inc} = k(\gamma_T[\cos\theta - \beta_T], \sin\theta, 0)$$

and, as a consequence, I may write κ_{inc} as

$$\kappa_{inc} = k\left(\gamma_T(1 - \beta_T\cos\theta),\ \gamma_T(\cos\theta - \beta_T),\ \sin\theta,\ 0\right) \quad .$$

I can also compute $\underline{k}_{refl} = \underline{k}_{inc} - 2(\underline{k}_{inc} \cdot \underline{n})\,\underline{n}$,

$$\underline{k}_{refl} = k(\gamma_T[\beta_T - \cos\theta], \sin\theta, 0) \quad .$$

Whence,

$$\kappa_{refl} = k\left(\gamma_T(1 - \beta_T\cos\theta),\ \gamma_T(\beta_T - \cos\theta),\ \sin\theta,\ 0\right) \quad .$$

Putting all of this into Equation (7) yields κ',

$$\kappa' = k\gamma_T^2\left(1 - 2\beta_T\cos\theta + \beta_T^2,\ 2\beta_T - (1 + \beta_T^2)\cos\theta,\ \gamma_T^{-2}\sin\theta,\ 0\right) \quad .$$

This I need to compare with $\kappa = k(1, \cos\theta, \sin\theta, 0) = (k_0, \underline{k})$.

The Doppler shift in frequency is $c(k_0 - k'_0)$ or

$$c(k_0 - k'_0) = c\,[k - k\gamma_T^2(1 - 2\beta_T\cos\theta + \beta_T^2)] = 2ck\beta_T\gamma_T^2(\cos\theta - \beta_T)$$

i.e.,

$$\omega' = \frac{\omega(1 - 2\beta_T\cos\theta + \beta_T^2)}{1 - \beta_T^2} \quad .$$

If the photon is normally incident ($\theta = 0$) then this takes a particularly simple form;

$$\omega'(\theta = 0) = \omega\left(\frac{1 - \beta_T}{1 + \beta_T}\right) \simeq \omega(1 - 2\beta_T) \text{ as } \beta_T \to 0 \quad .$$

This should be a familiar formula from monostatic radar work. Astronomers are used to seeing only one factor of β_T when dealing with celestial sources. That is owing to the fact that in such work only a single Lorentz transformation is involved, i.e., into the receiver's reference frame. In this instance I have gone back and forth yielding two factors of beta.

The antithesis of normal incidence is grazing incidence by a ray parallel to the satellite's surface. Now $\theta = \pi/2$ and

$$\omega'(\theta = \pi/2) = \omega[(1 + \beta_T^2)/(1 - \beta_T^2)]$$

an inherently relativistic effect because it is second order.

To see the effects of aberration, consider the reflection process in the original instantaneously inertial, heliocentric reference frame. Then, Figure 1 looks like Figure 2. While θ is both the angle of incidence and the angle between $\underline{k}$ and $\underline{\beta}_T$, θ' is the angle of reflection while $\pi - \theta'$ is the angle between $\underline{k}'$ and $\underline{\beta}_T$. Therefore,

$$\tan(\pi - \theta') = \frac{\gamma_T^{-2} \sin\theta}{-[-2\beta_T + (1 + \beta_T^2)\cos\theta]}$$

or

$$\tan\theta' = \frac{(1 - \beta_T^2)\sin\theta}{(1 + \beta_T^2)\cos\theta - 2\beta_T} \quad .$$

This can be rewritten as

$$\tan(\theta'/2) = [(1 + \beta_T)/(1 - \beta_T)]\tan(\theta/2) \quad .$$

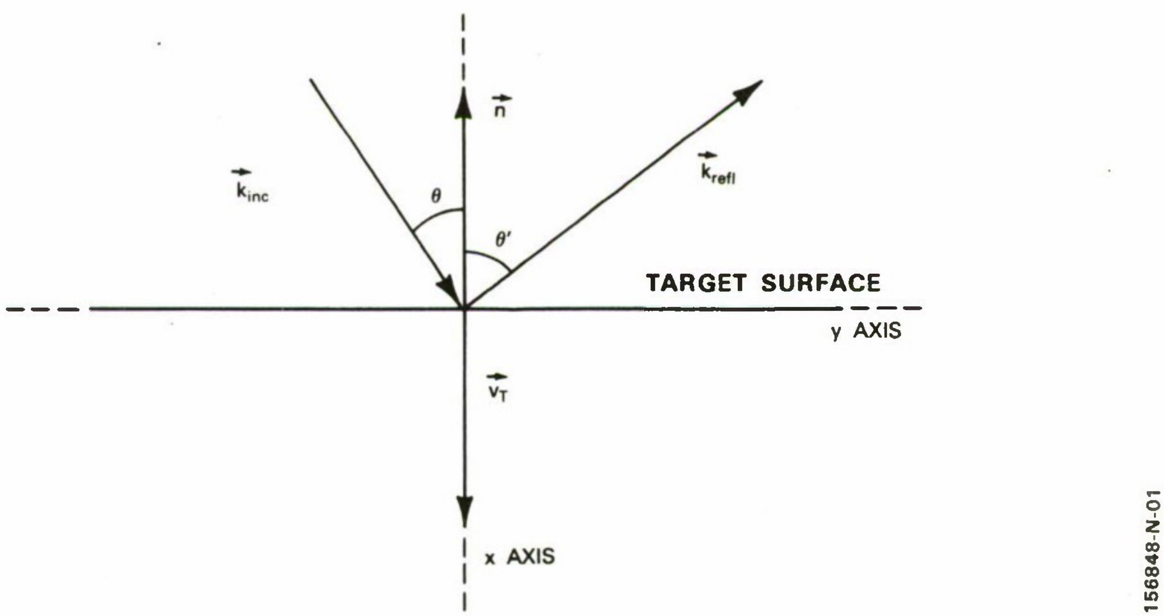

Figure 2. Reflection process in the inertial reference frame of the emitter.

Consider again the case of normal incidence; if $\theta = 0$, then $\theta' = 0$. This is a simple statement of the fact that the wave vectors are always perpendicular to the wavefronts. In the case of grazing incidence,

$$\tan \theta' = -(1 - \beta_T^2)/(2\beta_T)$$

which shows that for this to occur, it must be that $\beta_T < 0$ when $\theta' \in [0, \pi/2]$.

Now let the sensor move too, $\underline{\beta}_S = (\beta_S, 0, 0)$. The computation of κ_{inc} and κ_{refl} is unchanged. From Equation (3), I deduce $\underline{\beta}'_S$,

$$\underline{\beta}'_S = \left(\frac{\beta_S - \beta_T}{1 - \beta_S\beta_T}\right)(1, 0, 0) \quad .$$

Whence, if $\underline{k}_{refl}$ is the space-like part of κ_{refl} [i.e., $\underline{k}_{refl} = \underline{k}_{inc} - 2(\underline{k}_{inc} \cdot \underline{n})\, \underline{n}$], then the sensor detects an original photon of wave vector κ as one of wave vector κ' after reflection,

$$\kappa' = \Big(\gamma'_S\{k\gamma_T(1 - \beta_T \cos\theta) - \underline{\beta}'_S \cdot \underline{k}_{refl}\},$$
$$\underline{k}_{refl} + [(\gamma'_S - 1)\underline{\beta}'_S \cdot \underline{k}_{refl}/(\beta'_S)^2]\underline{\beta}'_S - \gamma'_S\gamma_T k(1 - \beta_T \cos\theta)\underline{\beta}'_S\Big) \quad .$$

A photon coming directly toward the sensor has a perceived wave vector of κ'' [see Equation (8)],

$$\kappa'' = \Big(\gamma_S k(1 - \beta_S \cos\theta),\ \underline{k} + [(\gamma_S - 1)\underline{\beta}_S \cdot \underline{k}/\beta_S^2]\underline{\beta}_S - \gamma_S k \underline{\beta}_S\Big) \quad .$$

The Doppler shift $c(k''_0 - k'_0)$ is given by

$$c(k''_0 - k'_0) = ck\{\gamma_S(1 - \beta_S \cos\theta) - \gamma'_S\gamma_T[1 - \beta_T \cos\theta - \beta'_S(\beta_T - \cos\theta)]\} \quad .$$

As $c \to \infty$ the Doppler shift approaches, including all β^2 terms, $2ck(\beta_T - \beta_S)(\cos\theta - \beta_T)$

or

$$\omega' = \omega \frac{[1 - 2\beta_T \cos\theta + \beta_T^2 - \beta_S(2\beta_T - \cos\theta - \beta_S/2)]}{1 - \beta_T^2} \quad .$$

Similarly, the aberration is given by

$$\tan\theta' = \frac{(\gamma'_S)^{-1}\,\gamma_T^{-1} \sin\theta}{(1 - \beta'_S\beta_T)\cos\theta - \beta_T + \beta'_S}$$

where $\beta'_S = (\beta_S - \beta_T)/(1 - \beta_S\beta_T)$. As $c \to \infty$, through all β^2 terms, this reduces to

$$\tan\theta' \simeq \frac{(1 - \beta_T^2)\sin\theta - \beta_S(\beta_S/2 - \beta_T)\sin\theta}{(1 + \beta_T^2)\cos\theta - 2\beta_T + \beta_S(1 - \beta_T\cos\theta)} \quad .$$

VII. THE "DOPPLER" SHIFT

A. THE REFERENCE FRAME

The analysis presented in Section V is an instantaneous one in that it deals with a particular photon emitted by the Sun at a particular instant of time. One might argue that to close the analysis the sensor should compare κ' in Equation (7) with κ'', the wave vector of a photon of original wave vector κ emitted by the Sun. And, in Subsection B below, this is what I shall do. However, the reader should be aware of the facts that neither κ' nor κ'' are compelling choices. Indeed, rather than starting with κ, that is an instantaneous value, it might be preferable to use an averaged emitted solar spectrum. One might average over a month to smooth out the lunar perturbations on the Earth's orbit about the Earth-Moon barycenter. Alternatively, a longer time span, such as a year, might be utilized to smooth out the reflection of the solar motion about the Earth-Moon barycenter. Or, we might average over a Jovian sidereal period of revolution and thereby smooth out the remaining major planetary perturbations contributing to the Sun's motion about the solar system barycenter. (I am not neglecting the contributions of Venus and Mars; Jupiter's period is sufficiently long that averaging over it effectively smooths their perturbations too.) No high precision solar spectrum has been observed over a long-enough time interval to provide the data necessary for such averaging.

Now consider the comparison for κ'. Should it be κ itself? If I use κ'', the wave vector perceived by the sensor directly, then how do we compute κ''? The difficulty is that, in general, a photon that left the Sun at $t = 0$, traveled along a geodesic to the target satellite, and thence along another geodesic to the sensor, required a different flight time than a photon that traveled directly to the sensor along a third geodesic. The Sun undergoes accelerated motion so that photons emitted at different times with the same wave vector relative to their instantaneously inertial heliocentric reference frames have different wave vectors in each others' reference frames. I must either show that the difference is of the second order for this analysis (as it turns out to be), correct for the effect, or retreat to one of the averaged solar spectrums referred to above.

There are more complications. While κ'' is the natural wave vector for the sensor to use as a comparison, assuming that a consistent reference frame for the two κ's can be found, the sensor is moving relative to the center of mass of the Earth. The target satellite's or missile's motion is most easily analyzed in a geocentric reference frame. Thus, a comparison of κ'' to κ' does not immediately yield a geocentric kinematical variable (or combination thereof). If I throw the luni-solar perturbations of the Earth's motion back into the discussion, then the potential for complication will increase further.

My principal purpose in recounting the above list is to focus attention on the nature of the problem. The ultimate best coordinate system is not yet clear. The analysis contained herein though is both complete and rigorous so that any minor modifications can be applied in a straightforward manner.

B. THE EXACT RESULT

Sloughing over for the moment the exact description of the reference frame of emission, consider a photon with wave vector κ relative to the Sun that travels directly to the sensor. Further suppose that it arrives at the sensor at time t_S as measured in the original instantaneously inertial, heliocentric reference frame. How will an observer at rest with respect to the sensor perceive it? Because the velocity of the sensor, relative to the original, instantaneously inertial, heliocentric reference frame is $\underline{v}_S$ at time $t = t_S$, the perceived wave vector κ'' is just [see Equation (1c)]

$$\kappa'' = (k_0'', \underline{k}'') = \left(\gamma_S(k_0 - \underline{k} \cdot \underline{\beta}_S),\ \underline{k} + [(\gamma_S - 1)\,\underline{k} \cdot \underline{\beta}_S/\beta_S^2]\,\underline{\beta}_S - \gamma_S k_0 \underline{\beta}_S\right) \qquad (8)$$

where $\underline{\beta}_S = \underline{v}_S/c$, $\gamma_S = (1 - \beta_S^2)^{-1/2}$.

C. THE POST-NEWTONIAN APPROXIMATION

Because $\beta \simeq 10^{-4}$ for all objects in all the reference frames under consideration, it makes sense to consider the limits of Equations (7,8) as $c \rightarrow \infty$. In this instance, all the gammas approach unity so

$$\kappa' \rightarrow \left(k_0 - \underline{k} \cdot \underline{\beta}_T - [\underline{k}_{inc} - 2(\underline{k}_{inc} \cdot \underline{n})\,\underline{n}] \cdot \underline{\beta}_S' ,\quad \underline{k}_{inc} - 2(\underline{k}_{inc} \cdot \underline{n})\,\underline{n} - (k_0 - \underline{k} \cdot \underline{\beta}_T)\,\underline{\beta}_S'\right)$$

$$\kappa'' \rightarrow (k_0 - \underline{k} \cdot \underline{\beta}_S,\ \underline{k} - k_0 \underline{\beta}_S)$$

and

$$k_0'' - k_0' \rightarrow -(\underline{\beta}_S - \underline{\beta}_T) \cdot \underline{k} + [\underline{k}_{inc} - 2(\underline{k}_{inc} \cdot \underline{n})\,\underline{n}] \cdot \underline{\beta}_S' \quad .$$

From Equation (6) we can see that

$$\underline{\beta}_S' \rightarrow \underline{\beta}_S - \underline{\beta}_T$$

which is the Newtonian result. From Equation (5b) the form for $\underline{k}_{inc}$ reduces to

$$\underline{k}_{inc} \rightarrow \underline{k} - k\underline{\beta}_T$$

whence

$$k_0'' - k_0' \rightarrow -2[(\underline{\beta}_S - \underline{\beta}_T) \cdot \underline{n}]\,[\underline{k} \cdot \underline{n}] \quad . \qquad (9)$$

This result is not transparent, and I now turn to attempting to make it at least translucent.

D. THE CLASSICAL FORMULA*

During the time interval from $t = 0$ until $t = t_T$, the incident photon travels from where the Sun was at $t = 0$ in the solar system barycentric inertial reference frame to where the target will

* This form of the analysis was suggested by E.J. Kelly and was developed further by R.C. Raup.

be at $t = t_T$. The target's geocentric location at time t_T is $\underline{r}_T$, while the Sun's geocentric location (a different geocenter!) at 0 is $\underline{S}_0$. Thus,

$$ct_T = |\underline{S}_0 - \underline{r}_T| = |\underline{S}(0) - \underline{r}(t_T)| \quad .$$

Similarly, the time interval $t_S - t_T$ represents the reflected photon's flight time from the target to the sensor. The sensor's geocentric (a third geocenter) location when the reflected photon arrives will be $\underline{\rho}_S$. Therefore,

$$c(t_S - t_T) = |\underline{\rho}_S - \underline{r}_T| = |\underline{\rho}(t_S) - \underline{r}(t_T)| \quad .$$

These two expressions can be generalized by writing them as

$$ct_T = |\underline{S}(t - t_S) - \underline{r}[t - (t_S - t_T)]|_{t=t_S} \tag{10a}$$

$$c(t_S - t_T) = |\underline{\rho}(t) - \underline{r}[t - (t_S - t_T)]|_{t=t_S} \quad . \tag{10b}$$

Because the sensor, target, Earth, and Sun are all in motion, both t_S and t_T are implicitly dependent upon solar system barycentric inertial proper time too. The Doppler shifts induced by these motions may be obtained by differentiating Equations (10a) and (10b) with respect to t and then adding. Keeping the implicit dependence of t_S and t_T on t in mind, the results are, after evaluation at $t = t_S$,

$$c\dot{t}_T = \left\{(1 - \dot{t}_S)\dot{\underline{S}}_0 - [1 - (\dot{t}_S - \dot{t}_T)]\dot{\underline{r}}_T\right\} \cdot \left\{\underline{S}_0 - \underline{r}_T\right\}/(ct_T) \tag{11a}$$

$$c(\dot{t}_S - \dot{t}_T) = \left\{\dot{\underline{\rho}}_S - [1 - (\dot{t}_S - \dot{t}_T)]\dot{\underline{r}}_T\right\} \cdot \left\{\underline{\rho}_S - \underline{r}_T\right\}/[c(t_S - t_T)] \quad . \tag{11b}$$

With three approximations this will reduce to the naive Newtonian viewpoint that $c\dot{t}_S$ should represent the instantaneous time rate of change of the Sun-target-sensor distance.

First note that $c\dot{t}_S$ and $c\dot{t}_T$ are small, on the order of 1 in 10^4 (e.g., $\simeq 30$ km/s/c). Hence, their presence on the right-hand sides of Equations (11) may be ignored. Second, observe that the principal motion of the Sun about the solar system barycenter is owing to Jupiter's attraction. This orbit has an amplitude of at most $M_{♃}/M_{\odot} \simeq 5 \times 10^{-3}$ A.U. and, in fact $\simeq 5 \times 10^{-6}$ A.U. The period is $\simeq 12$ yr with a linear tangential speed of $\simeq 1$ m/s. Therefore, since $t_T \simeq 500$ s, I can ignore the difference among $\underline{S}_0$, $\underline{S}_T$, and $\underline{S}_S$ as well as among the corresponding velocities. Third, note that $t_S - t_T \simeq 0.02$ s, so I may approximate $\dot{\underline{\rho}}_S$ and $\underline{\rho}_S$ by $\dot{\underline{\rho}}_T$ and $\underline{\rho}_T$. As $\underline{\rho}$ and $\dot{\underline{\rho}}$ are known, I can repair this approximation as an additional term in the planetary aberration. Thus, the sum of Equations (11) can be approximately written, with all now superfluous subscripts suppressed, as

$$c\dot{t} \simeq (\dot{\underline{S}} - \dot{\underline{r}}) \cdot \frac{(\underline{S} - \underline{r})}{|\underline{S} - \underline{r}|} + (\dot{\underline{\rho}} - \dot{\underline{r}}) \cdot \frac{(\underline{\rho} - \underline{r})}{|\underline{\rho} - \underline{r}|} \quad . \tag{12}$$

It is also true that ρ/S and r/S are $\simeq 5 \times 10^{-5}$. So, with $\underline{s} = \underline{S}/S$ and only terms in $1/S$ kept, the final expression is (good to 1 part in 10^4)

$$V_R' = c\dot{t} = \dot{R} + \underline{s} \cdot (\dot{\underline{S}} - \dot{\underline{r}}) + \underline{s} \cdot (\dot{\underline{S}} - \dot{\underline{r}})(\underline{r} \cdot \underline{s}/S) \; - \underline{r} \cdot (\dot{\underline{S}} - \dot{\underline{r}})/S$$

where $R = |\underline{\rho} - \underline{r}|$ is the target-sensor distance. Through the use of $\underline{s} \cdot \underline{s} = 1$ and the vector identity $(\underline{a} \times \underline{b}) \cdot (\underline{c} \times \underline{d}) = (\underline{a} \cdot \underline{c})(\underline{b} \cdot \underline{d}) - (\underline{a} \cdot \underline{d})(\underline{b} \cdot \underline{c})$, this can be recast as

$$V_R' = \dot{R} + \underline{s} \cdot (\dot{\underline{S}} - \dot{\underline{r}}) + [(\dot{\underline{S}} - \dot{\underline{r}}) \times \underline{s}] \cdot (\underline{s} \times \underline{r}/S) \quad . \tag{13}$$

What is the relationship between Equations (12) and (9)? Equation (9) is the nonrelativistic limit of the fully covariant expression for $k_0'' - k_0'$. It includes the unknown outward surface normal of the target at the time of reflection (e.g., $\underline{n}$ at time $t = t_T$). It also utilizes velocity vectors referred to the original (i.e., $t = 0$), instantaneously inertial, heliocentric reference frame. Similarly, the space-like part of the original wave vector also enters in Equation (9). However, in Equation (12) we see geocentric locations for the target ($\underline{r}$), the sensor ($\underline{\rho}$), and the Sun ($\underline{S}$). As my earlier Doppler-shift results were all expressed in the form $\omega' = \omega(1 + \text{correction})$, I shall repeat this form herein.

The original photon traveled along $\underline{d}_T$, the location of the target at time $t = t_T$ in the instantaneously inertial, heliocentric reference frame of $t = 0$. Therefore,

$$\underline{k} = k\, \underline{d}_T / d_T \quad .$$

The incoming wave vector κ_{inc}, see Equation (5b), is a Lorentz-transformed version of $\kappa = (k_0, \underline{k})$. It is turned into κ_{refl}, see Equation (5a), and then travels to the sensor. Thus, $\underline{k}_{refl}$ travels from $\underline{d}_T$ to $\underline{d}_S$, the location of the sensor in the instantaneously inertial, $t = 0$ heliocentric reference frame. It follows that I can write $\underline{k}_{refl}$ as

$$\underline{k}_{refl} = k_{refl}(\underline{d}_S - \underline{d}_T)/|\underline{d}_S - \underline{d}_T| \quad .$$

When I have done this, to lowest order in β, I can derive an expression that eliminates $\underline{n}$.

To start with,

$$\kappa = (k_0, \underline{k}) = k(1, \underline{d}_T/d_T)$$

$$\kappa_{inc} \xrightarrow[\beta_T \to 0]{} k(1 - \underline{d}_T \cdot \underline{\beta}_T/d_T,\ \underline{d}_T/d_T - \underline{\beta}_T) \quad .$$

Now, it follows that

$$\kappa_{refl} \to k\Big(1 - \underline{d}_T \cdot \underline{\beta}_T/d_T,\ \underline{d}_T/d_T - \underline{\beta}_T - 2[(\underline{d}_T/d_T - \underline{\beta}_T) \cdot \underline{n}]\,\underline{n}\Big)$$

$$= \Big(k(1 - \underline{d}_T \cdot \underline{\beta}_T/d_T),\ k_{refl}(\underline{d}_S - \underline{d}_T)/|\underline{d}_S - \underline{d}_T|\Big) \quad .$$

To lowest order,

$$k_{refl} = k(1 - \underline{d}_T \cdot \underline{\beta}_T/d_T)$$

so

$$(1 - \underline{d}_T \cdot \underline{\beta}_T/d_T)\, \frac{(\underline{d}_S - \underline{d}_T)}{|\underline{d}_S - \underline{d}_T|} = \underline{d}_T/d_T - \underline{\beta}_T - 2\,[(\underline{d}_T/d_T - \beta_T) \cdot \underline{n}]\,\underline{n} \quad .$$

From Equation (9) the quantity needed is $[(\underline{\beta}_S - \underline{\beta}_T) \cdot \underline{n}]\,(\underline{k} \cdot \underline{n})$ or $k(\underline{n} \cdot \underline{d}_T/d_T)\,[(\underline{\beta}_S - \underline{\beta}_T) \cdot \underline{n}]$. By forming the scalar product of the above expression with $\underline{\beta}_S - \underline{\beta}_T$ and keeping only the first-order terms, the equality

$$(\underline{n} \cdot \underline{d}_T/d_T)\,[(\underline{\beta}_S - \underline{\beta}_T) \cdot \underline{n}] = \tfrac{1}{2}(\underline{\beta}_S - \underline{\beta}_T) \cdot [\underline{d}_T/d_T - (\underline{d}_S - \underline{d}_T)/|\underline{d}_S - \underline{d}_T|\,]$$

follows.

The next task is to express $\underline{d}_S$ and $\underline{d}_T$ in terms of the geocentric quantities $\underline{\rho}_S$ and $\underline{r}_T$. When done properly, the kind of implicit time dependence seen earlier in this Subsection appears. As my intent is only to derive Equation (9), and all the refinements are gone therein, I shall make the necessary approximations *ab initio*. Thus,

$$\underline{d}_S = \underline{\rho} - \underline{S} \quad , \quad \underline{d}_T = \underline{r} - \underline{S}$$

and

$$k_0' = k - \underline{k} \cdot \underline{\beta}_S - 2[(\underline{\beta}_S - \underline{\beta}_T) \cdot \underline{n}]\,(\underline{k} \cdot \underline{n})$$

so

$$k_0'/k \rightarrow 1 - \frac{(\dot{\underline{r}} - \dot{\underline{S}})}{c} \cdot \frac{(\underline{r} - \underline{S})}{|\underline{r} - \underline{S}|} - \frac{\dot{\underline{R}}}{c} \cdot \frac{\underline{R}}{R}$$

or

$$\omega' = \omega(1 - V_R'/c) \quad . \tag{14}$$

VIII. CELESTIAL MECHANICS IMPLICATIONS

From Equation (14) it is clear that the kinematical quantity that a radial velocity spectrometer will determine is V_R'. Looking at Equation (13) we see that V_R' involves $\underline{r}$ and $\dot{\underline{r}}$ in a complicated fashion. As $\underline{r}$ can be written as $\underline{r} = r\underline{\ell}$, where $\underline{\ell}$ is a unit vector in the direction of $\underline{r}$, it follows that $\dot{\underline{r}} = \dot{r}\underline{\ell} + r\dot{\underline{\ell}}$. Thus, since both $\underline{r}$ and $\dot{\underline{r}}$ are involved, the determination of V_R' provides information about the distance r, the geocentric radial velocity $\dot{r}$, the geocentric position $\underline{\ell}$, and the geocentric angular velocity $\dot{\underline{\ell}}$. Clearly, the incorporation of this type of information in a differential correction procedure represents no new conceptual problem or difficulty. On the other hand, the utilization of V_R' in an initial orbit determination scheme presents unique difficulties. A special algorithm is required to maximize the information content in V_R' (which will presumably be acquired along with the topocentric position and angular velocity). This algorithm awaits invention.

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This report provides the theoretical analysis, within the framework of special relativity, for the information content of a radial velocity spectrometer. This device is an automated telescope-photometer-spectrometer combination. It measures the displacement between a Doppler-shifted set of spectral lines and its at-rest version. Thus, it provides a measurement of "radial velocity." Exactly what kinematical content is revealed by such a measurement within the solar system is brought out herein. The discussion is couched in terms of a satellite-based sensor observing another orbiting body. Both objects are presumed to orbit about the Earth. Sufficient generality is presented to treat any other case also. In the specific case of interest, both the topocentric distance and the topocentric radial velocity are present in the measured quantity.